Composable Uncertainty in Symmetric Monoidal Categories for Design Problems

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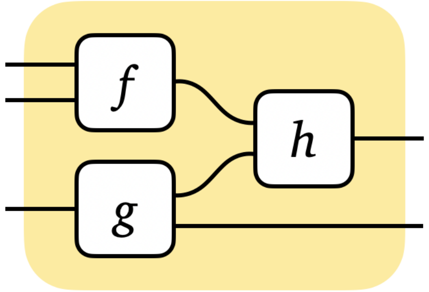
SMCs with extra structure as system models

(symmetric) monoidal



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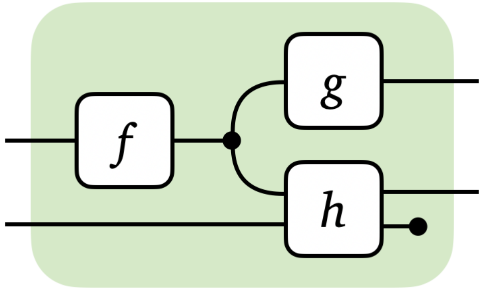
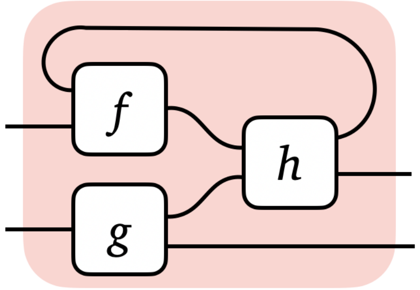
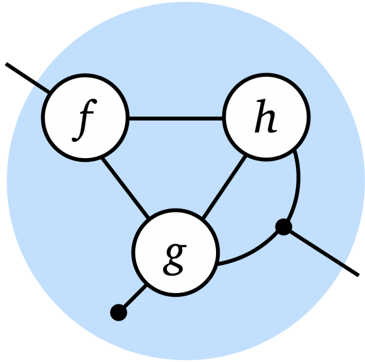


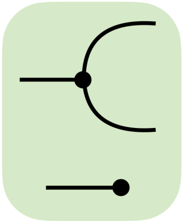
compact closed

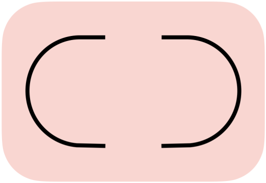


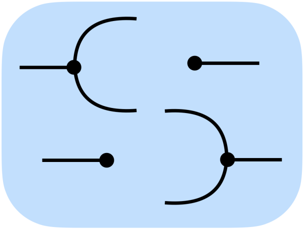
hypergraph



Extra structure:

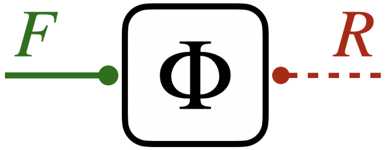






Examples:	FinStoch	DP	$\mathbf{Csp}(\mathsf{Petri}_r)$

Monotone co-design



A ***feasibility relation*** is a monotone map

$$\Phi \colon F^{\mathrm{op}} \times R \to \mathrm{Bool}$$

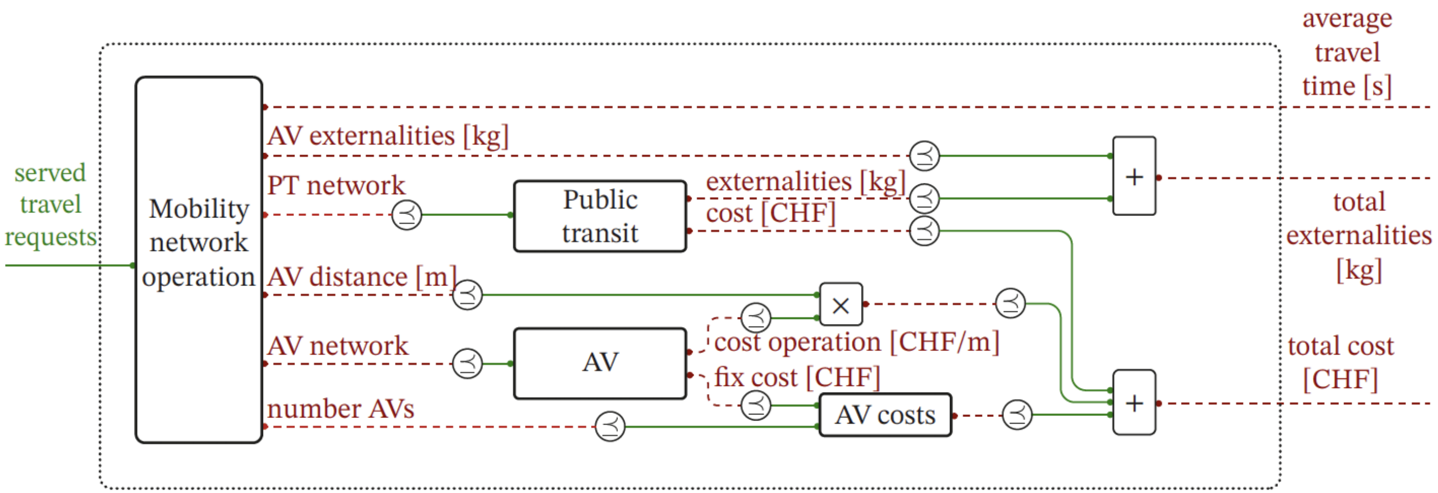
where Bool $:= \{\bot \leq \top\}$.

Posets and feasibility relations form ***compact closed SMC*** called DP.

DP can be viewed as ***enriched in*** Pos.

Functorial queries via $\mathrm{DP} \simeq \mathrm{Kl}_{\mathrm{U}}$ efficiently compute optimal designs.

Applied in real world problems.

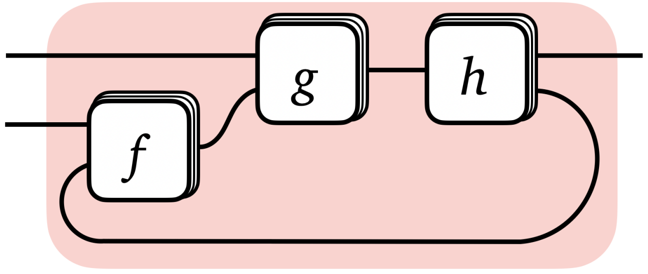


Sources of uncertainty:

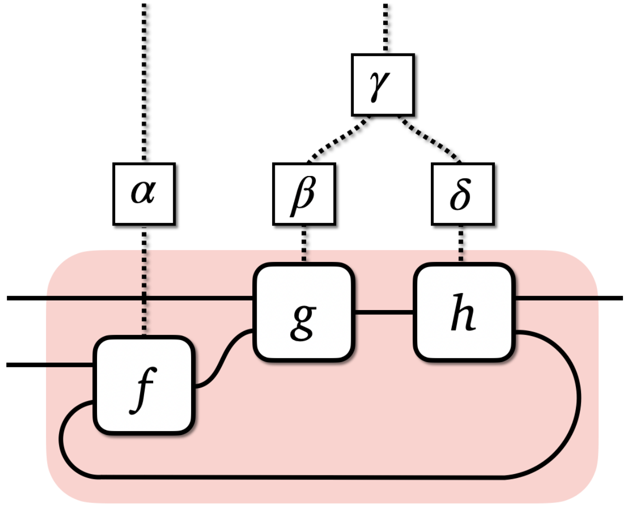
- Random performance
- Dependency on external factors
- Partial knowledge of feasibility

G. Zardini: *Co-Design of Complex Systems: From Autonomy to Future Mobility Systems.*

Uncertainty for SMCs



monadic uncertainty



parametric uncertainty

Symmetric monoidal $\mathcal{V}$-category $\mathcal{C}$

$+$

Symmetric monoidal monad (M, η, μ, ∇) on $\mathcal{V}$

$=$

Symmetric monoidal $\mathcal{V}$-category $M_*\mathcal{C}$ with hom-objects $M\mathcal{C}(X, Y)$

- Functorial
- Transfers extra structure

Symmetric monoidal $\mathcal{W}$-category $\mathcal{C}$

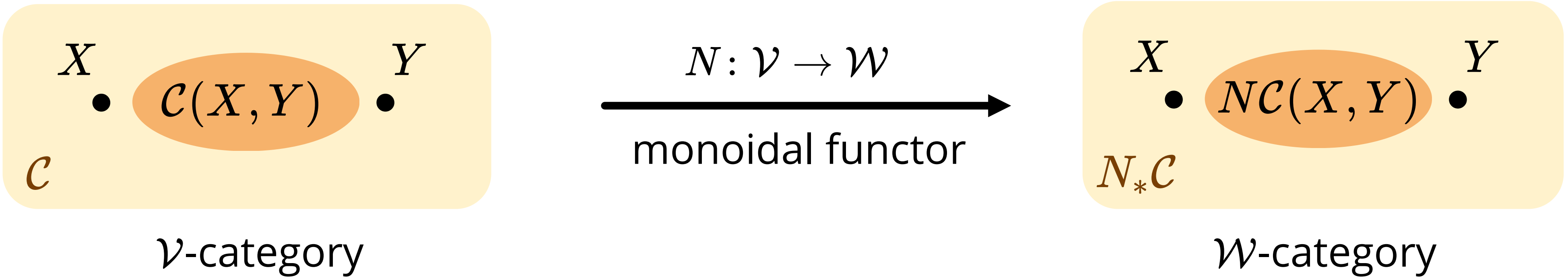
$+$

Full strict monoidal subcategory $i : \mathcal{U} \hookrightarrow \mathcal{W}$

$=$

Symmetric monoidal 2-category $\mathcal{U}/(-)_*\mathcal{C}$ with 1-cells $U \to \mathcal{C}(X, Y)$ in $\mathcal{W}$ and reparameterization 2-cells.

Change-of-base replaces hom-objects



$\mathcal{C}$ (symmetric) monoidal $\mathcal{V}$-category

N symmetric monoidal $\longrightarrow$

$N_*\mathcal{C}$ (symmetric) monoidal $\mathcal{W}$-category

$+$

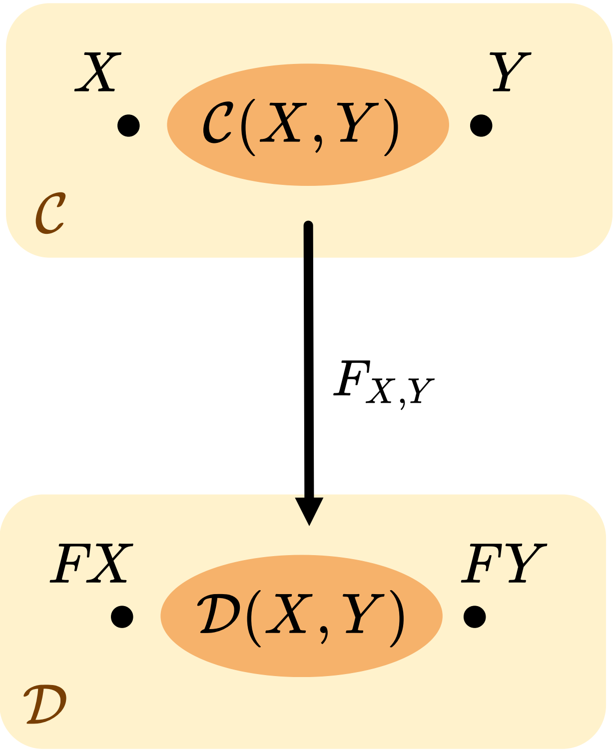
$\iota\colon \mathcal{C}_0 \to (N_*\mathcal{C})_0$ strict (symmetric) monoidal

Enriched category theory

Definition: Given an SMC $(\mathcal{V}, \otimes, I)$, a ***$\mathcal{V}$-category*** $\mathcal{C}$ consists of

1. a set of *objects* $\mathsf{Ob}(\mathcal{C})$,
2. *hom-objects*
$$\mathcal{C}(A, B) \in \mathsf{Ob}(\mathcal{V}),$$
3. *composition* arrows
$$c_{A,B,C} : \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \to \mathcal{C}(A, C),$$
4. and *identity* arrows
$$\mathrm{id}_A : I \to \mathcal{C}(A, A),$$

satisfying unitality and associativity conditions.



Definition: A ***$\mathcal{V}$-functor*** $F : \mathcal{C} \to \mathcal{D}$ consists of

1. a function $F : \mathsf{Ob}(\mathcal{C}) \to \mathsf{Ob}(\mathcal{D})$,
2. $\mathcal{V}$-arrows $F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$

that preserve composition and identities.

Definition: A ***$\mathcal{V}$-natural transformation*** $\tau : F \Rightarrow G$ consists of components

$$\tau_A : I \to \mathcal{D}(FA, GA)$$

that satisfy the $\mathcal{V}$-naturality condition.

Monoidal functors

Definition: Let $(\mathcal{V}, \otimes, I)$ and $(\mathcal{W}, \bullet, J)$ be SMCs. A ***monoidal functor*** is a functor $N\colon \mathcal{V} \to \mathcal{W}$, along with natural transformations whose components

$$N_\epsilon \colon J \to NI$$

$$\widetilde{N}_{A,B} \colon NA \bullet NB \to N(A \otimes B)$$

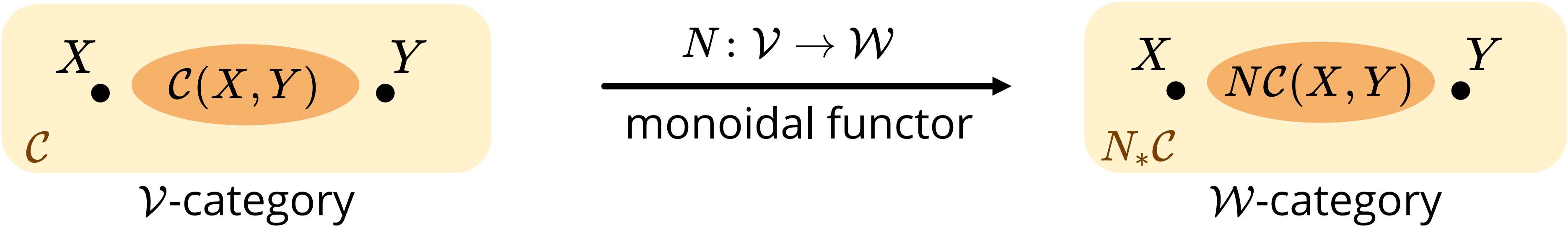
satisfy associativity and unitality conditions.

We call N ***strict***, if N_ϵ and $\widetilde{N}_{A,B}$ are identities.

We call N ***symmetric monoidal,*** *if* N preserves the symmetry coherence:

$$\begin{array}{ccc} NX \bullet NY & \xrightarrow{\sigma_W} & NY \bullet NX \\ \downarrow{\scriptstyle \widetilde{N}} & & \downarrow{\scriptstyle \widetilde{N}} \\ N(X \otimes Y) & \xrightarrow[N(\sigma_V)]{} & N(Y \otimes X) \end{array}$$

Change-of-base



Proposition *(Eilenberg & Kelly 1966):*

Any ***monoidal functor*** $N: \mathcal{V} \to \mathcal{W}$ induces a ***change-of-base*** 2-functor $N_*: \mathbb{V}\mathrm{cat} \to \mathbb{W}\mathrm{cat}$ that sends a $\mathcal{V}$-category $\mathcal{C}$ to the $\mathcal{W}$-category $N_*\mathcal{C}$ with

1. objects $\mathsf{Ob}(N_*\mathcal{C}) := \mathsf{Ob}(\mathcal{C})$,
2. arrows $(N_*\mathcal{C})(A,B) := N\mathcal{C}(A,B)$,
3. composition

$$N\mathcal{C}(A,B) \otimes_{\mathcal{W}} N\mathcal{C}(B,C) \xrightarrow{\widetilde{N}} N(\mathcal{C}(A,B) \otimes_{\mathcal{V}} \mathcal{C}(B,C)) \xrightarrow{Nc} N\mathcal{C}(A,C),$$

4. identities

$$I_{\mathcal{W}} \xrightarrow{N_\epsilon} NI_{\mathcal{V}} \xrightarrow{N\mathsf{id}_A} N\mathcal{C}(A,A).$$

Transferring SMC structure

Definition: A ***(symmetric) monoidal $\mathcal{V}$-category*** $(\mathcal{C}, \boxtimes, J)$ is a $\mathcal{V}$-category $\mathcal{C}$ with

1. a $\mathcal{V}$-functor $\boxtimes \colon \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$,
2. a unit object $J \in \mathtt{Ob}(\mathcal{C})$,
3. $\mathcal{V}$-natural isomorphisms

$$a_{A,B,C} \colon I \to \mathcal{C}((A \boxtimes B) \boxtimes C, A \boxtimes (B \boxtimes C)), \qquad r_A \colon I \to \mathcal{C}(A \boxtimes J, A),$$

$$l_A \colon I \to \mathcal{C}(A \boxtimes J, A), \qquad (s_{A,B}) \colon I \to \mathcal{C}(A \boxtimes B, B \boxtimes A),$$

that satisfy the usual coherence axioms in the underlying category $\mathcal{C}_0$.

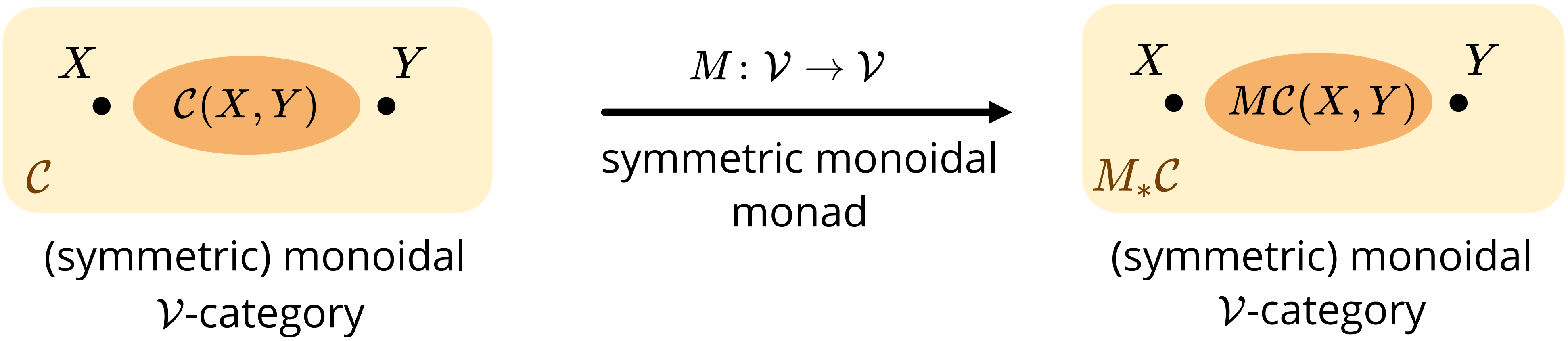
Proposition *(Cruttwell 2008):*

If $N \colon \mathcal{V} \to \mathcal{W}$ is a ***symmetric*** monoidal functor, and $\mathcal{C}$ a (symmetric) monoidal $\mathcal{V}$-category, then $N_*\mathcal{C}$ is a (symmetric) monoidal $\mathcal{W}$-category with product

$$N\mathcal{C}(A_1, B_1) \otimes_{\mathcal{W}} N\mathcal{C}(A_2, B_2) \xrightarrow{\widetilde{N}} N(\mathcal{C}(A_1, B_1) \otimes_{\mathcal{V}} \mathcal{C}(A_2, B_2)) \xrightarrow{N\boxtimes} N\mathcal{C}(A_1 \boxtimes A_2, B_1 \boxtimes B_2).$$

There is a strict (symmetric) monoidal functor $\iota \colon \mathcal{C}_0 \to (N_*\mathcal{C})_0$ between underlying categories.

Monadic uncertainty

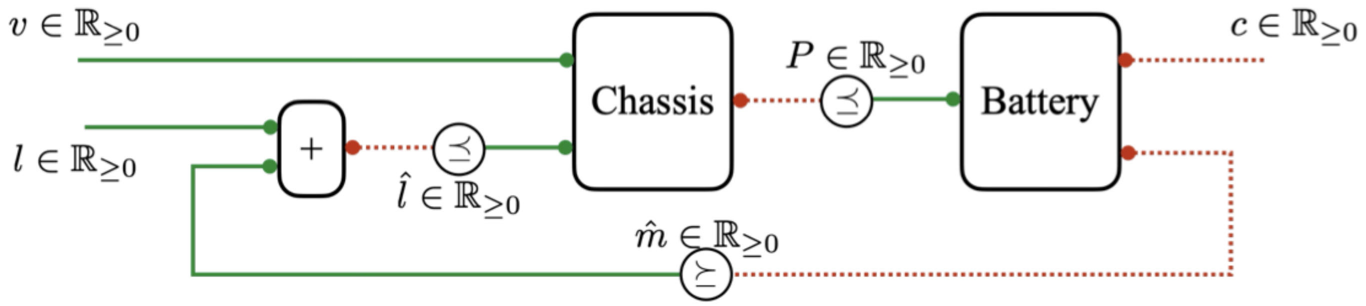


Definition: A ***symmetric monoidal monad*** (M, η, μ, ∇) on $\mathcal{V}$ consists of

1. a monad (M, η, μ) on $\mathcal{V}$,
2. a natural transformation $\nabla_{X,Y} : MX \otimes MY \to M(X \otimes Y)$

such that M is a symmetric monoidal functor with unit η_I, and μ, η are monoidal transformations.

Uncertain design problems



Non-empty powerset monad

$$\mathsf{Pow}_\emptyset : \mathsf{Set} \to \mathsf{Set}$$
$$X \mapsto \{A \subseteq X \mid A \neq \emptyset\}$$

Giry monad

$$\mathsf{D} : \mathsf{Meas} \to \mathsf{Meas}$$
$$X \mapsto \{\text{probability measures on } X\}$$

Interval monad

$$\mathsf{Int} : \mathsf{Pos} \to \mathsf{Pos}$$
$$(P, \leq) \mapsto \{[l, u] \mid l \leq u\} \subseteq P \times P$$

$$\eta_X : x \mapsto [x, x] \qquad \mu_X : [[l_1, u_1], [l_2, u_2]] \mapsto [l_1, u_2]$$

$$\nabla_{X,Y} : ([l_1, u_1], [l_2, u_2]) \mapsto [(l_1, l_2), (u_1, u_2)]$$

Lemma: There is a ***strict symmetric monoidal functor*** $\Sigma : \mathsf{Pos} \to \mathsf{Meas}$ that assigns each poset the σ-algebra generated by its upper sets.

Imprecise graphical models

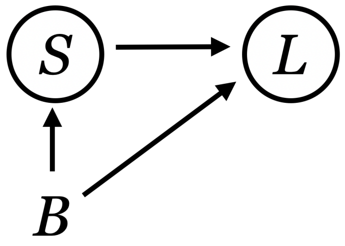
$(\mathsf{Pow}_{\emptyset})_*\mathsf{FinStoch}$ has

1. objects: Finite sets X, Y
2. arrows: Non-empty subsets S of Markov kernels $k\colon X \to Y$
3. composition:
$$T \circ S := \{t \circ s \mid t \in T, s \in S\}$$
4. identities:
$$\mathsf{id}_X = \{\mathsf{id}_X\}$$
5. monoidal product
$$S_1 \otimes S_2 := \{s_1 \otimes s_2 \mid s_1 \in S_1, s_2 \in S_2\}.$$

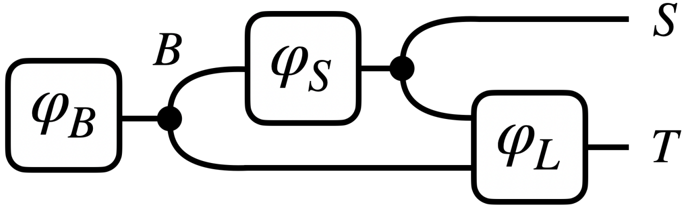
We transfer cd-structure by $\{\mathsf{cp}_X\}$ and $\{\mathsf{del}_X\}$ along the strict symmetric monoidal $\iota\colon \mathsf{FinStoch} \to M_*\mathcal{C}$

View probabilistic graphical models as string diagrams in FinStoch.

(Fritz & Klingler 2023, Lorenz & Tull 2023)



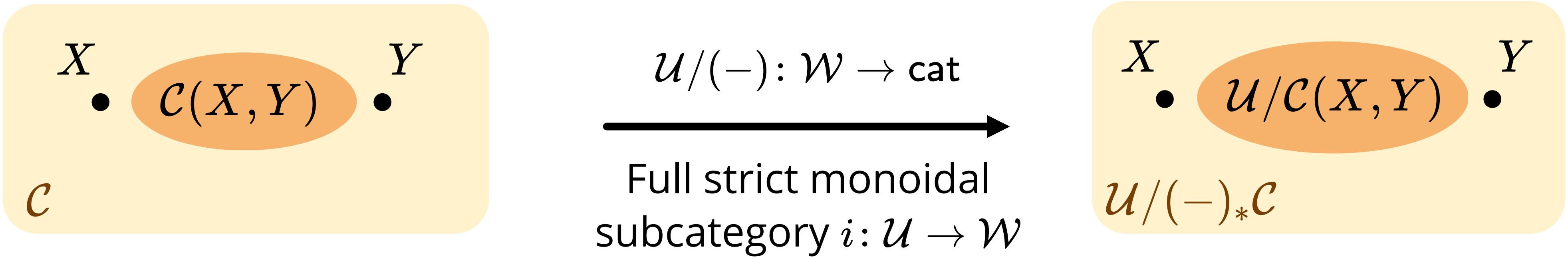
Bayesian network



String diagram in FinStoch

$\{\mathsf{del}_Y\} \circ S = \{\mathsf{del}_Y \circ s \mid s \in S\} = \{\mathsf{del}_X\}$ shows the result is even a Markov category.

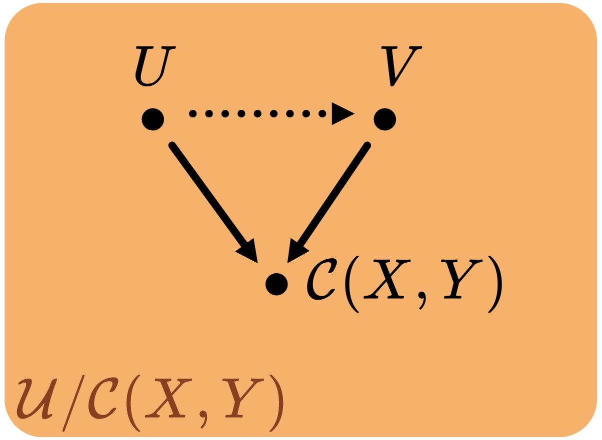
Parametric uncertainty



(symmetric) monoidal
$\mathcal{W}$-category

(symmetric) monoidal
2-category

slice category

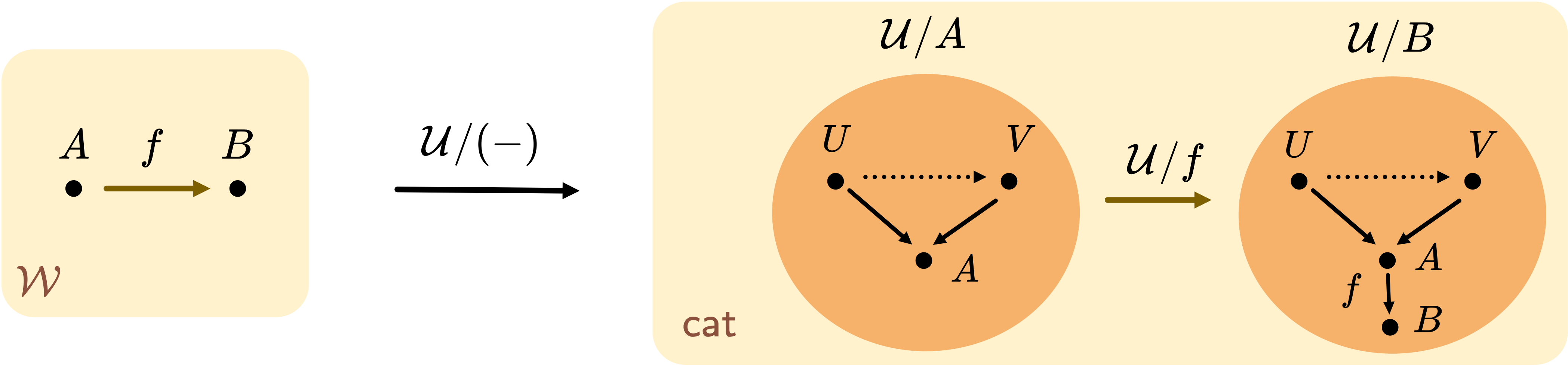


Partial slice functor

Definition: Given a category $\mathcal{W}$ and subcategory $i\colon \mathcal{U} \hookrightarrow \mathcal{W}$, the ***partial slice functor*** $\mathcal{U}/(-)\colon \mathcal{W} \to \mathsf{cat}$ sends $A \in \mathsf{Ob}(\mathcal{W})$ to the comma category $(i \downarrow \Delta_A)$ whose

1. objects are pairs $(U \in \mathsf{Ob}(\mathcal{U}), f\colon i(\mathcal{U}) \to A)$,
2. arrows $\varphi\colon (U, f) \to (V, g)$ are $\mathcal{U}$-arrows $\varphi\colon U \to V$ satisfying $f = g \circ i(\varphi)$.

On arrows, $\mathcal{U}/(-)$ sends $f\colon A \to B$ to the post-composition functor $\mathcal{U}/f : \mathcal{U}/A \to \mathcal{U}/B$.



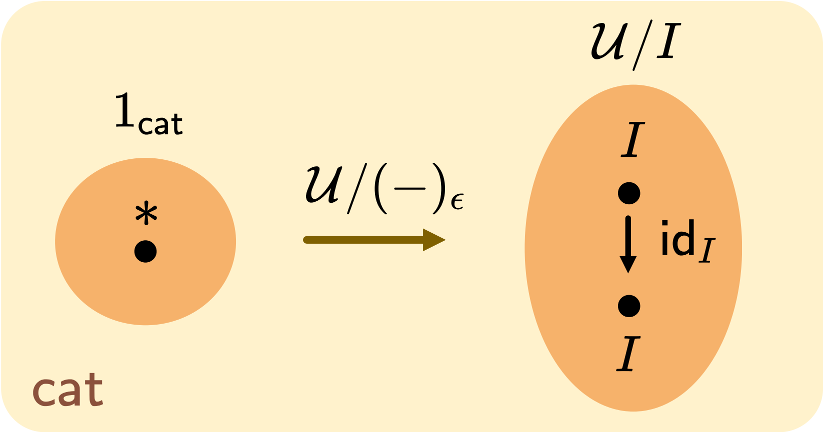
Partial slice functor

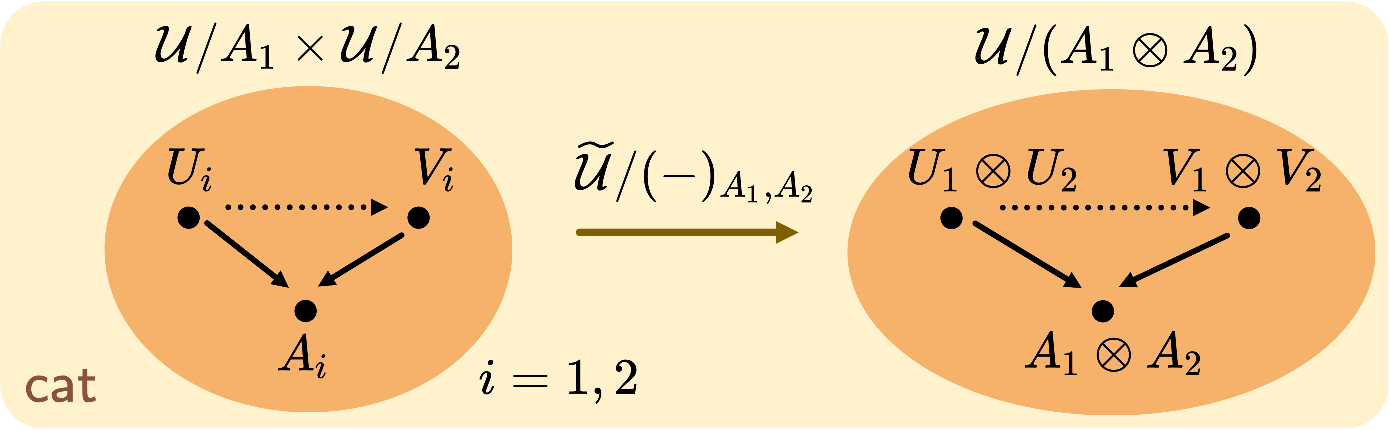
Proposition: Let $(\mathcal{W}, \otimes, I)$ be an SMC and $i\colon \mathcal{U} \hookrightarrow \mathcal{W}$ a full strict monoidal subcategory.

The ***partial slice functor*** $\mathcal{U}/(-)\colon \mathcal{W} \to$ cat ***is monoidal***:

1. The functor $\mathcal{U}/(-)_\epsilon\colon 1_{\text{cat}} \to \mathcal{U}/I$ maps the single object $*$ to the identity $\text{id}_I\colon I \to I$.
2. The natural transformation $\widetilde{\mathcal{U}}/(-)_{A_1,A_2}\colon \mathcal{U}/A_1 \times \mathcal{U}/A_2 \to \mathcal{U}/(A_1 \otimes A_2)$ maps
   - objects $f_i\colon U_i \to A_i$ to $f_1 \otimes f_2\colon U_1 \otimes U_2 \to A_1 \otimes A_2$
   - arrows $\varphi\colon f_1 \to g_1$ and $\psi\colon f_2 \to g_2$ to $\varphi \otimes \psi$.

$\mathcal{U}/(-)$ is only symmetric up to the natural invertible 2-cells σ_{U_1,U_2}.

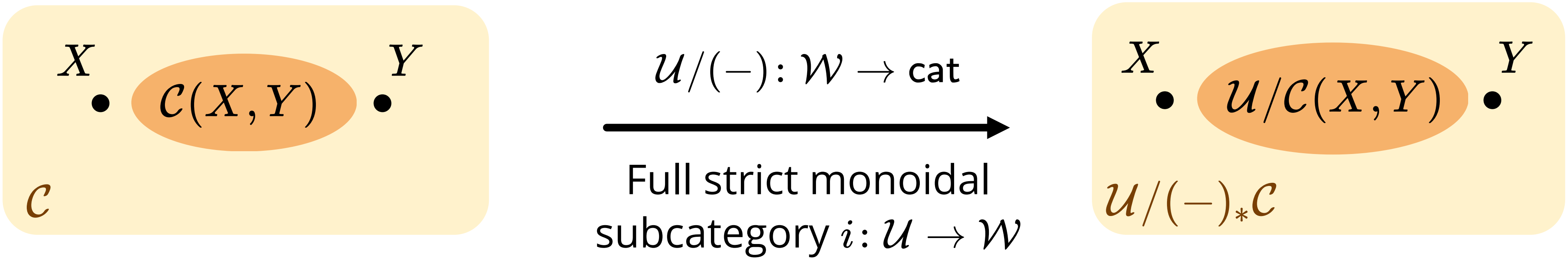




Parametric uncertainty

Theorem: Let $(\mathcal{C}, \boxtimes, J)$ be a (symmetric) monoidal $\mathcal{W}$-category and $i\colon \mathcal{U} \hookrightarrow \mathcal{W}$ a full strict monoidal subcategory.

1. $\mathcal{U}/(-)_*\mathcal{C}$ is a (symmetric) monoidal 2-category.
2. The inclusion 2-functor $\iota\colon \mathcal{C}_0 \to \mathcal{U}/(-)_*\mathcal{C}$ strictly preserves monoidal products and coherence 1-cells.



(symmetric) monoidal $\mathcal{W}$-category

(symmetric) monoidal 2-category

Parametric uncertainty

Theorem: Let $(\mathcal{C}, \boxtimes, J)$ be a (symmetric) monoidal $\mathcal{W}$-category and $i\colon \mathcal{U} \hookrightarrow \mathcal{W}$ a full strict monoidal subcategory.

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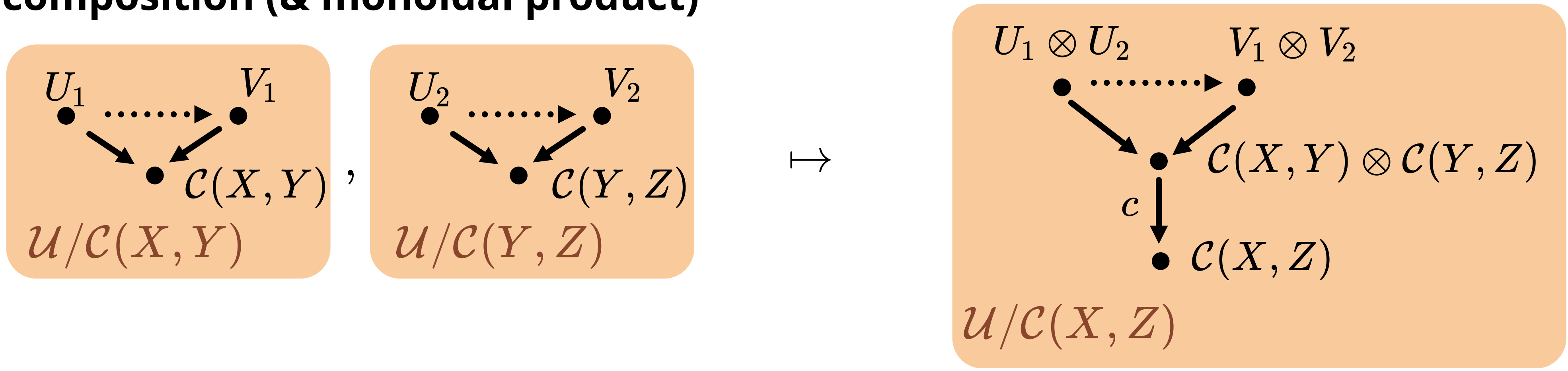
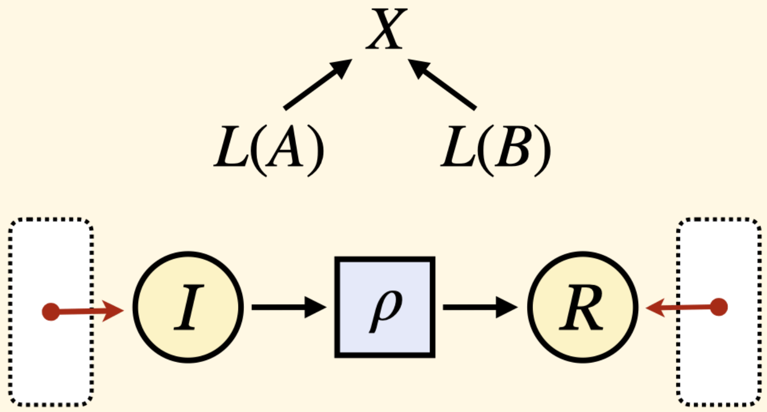
composition (& monoidal product)



Open reaction networks with coupled rates

Open Petri nets with rates: $\mathbf{Csp}(\mathsf{Petri}_r)$

(Baez & Courser 2020, Baez & Pollard 2017)

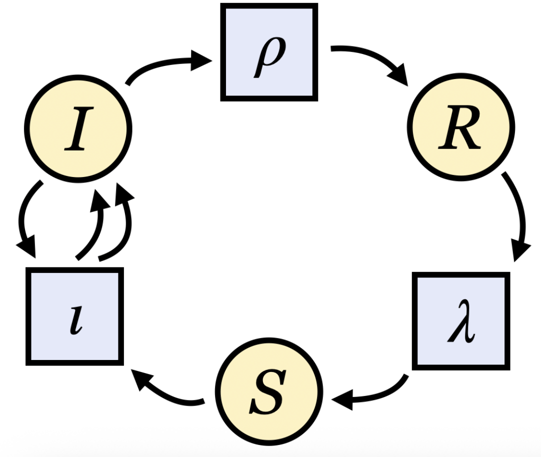


Put $\mathcal{C} := \mathbf{Csp}(\mathsf{Petri}_r)$ and $\mathcal{U}$ the full monoidal subcategory of Set generated by $\mathbb{R}_{\geq 0}$.

Then $\mathcal{U}/(-)_*\mathcal{C}$ has

1. objects: finite sets A, B
2. 1-cells: $f \colon \mathbb{R}^n_{\geq 0} \to \mathbf{Csp}(\mathsf{Petri}_r)(A, B)$
3. 2-cells: commuting $\varphi \colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}^m_{\geq 0}$

infection	**recovery**	**loss**
$\iota \colon S + I \xrightarrow{r_\iota} 2I$	$\rho \colon I \xrightarrow{r_\rho} R$	$\lambda \colon R \xrightarrow{r_\lambda} S$



$$\frac{dS}{dt} = r_\lambda R - r_\iota SI$$

$$\frac{dI}{dt} = r_\iota SI - r_\rho I$$

$$\frac{dR}{dt} = r_\rho I - r_\lambda R$$

We can transfer hypergraph structure via $\iota \colon f \mapsto [* \mapsto f]$.

Open reaction networks with coupled rates

Open Petri nets with rates: $\mathbf{Csp}(\mathsf{Petri}_r)$

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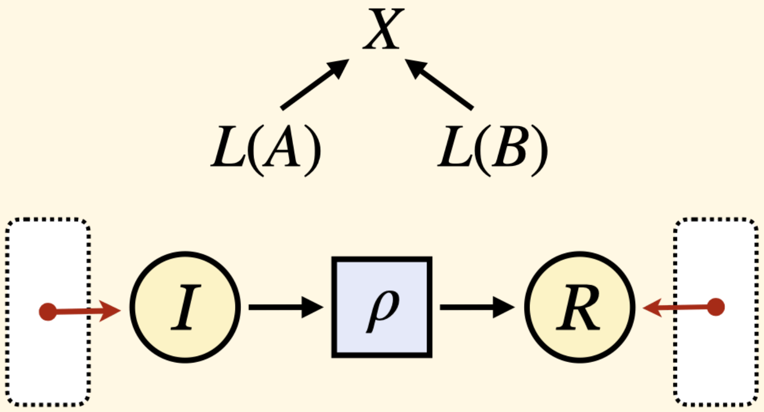

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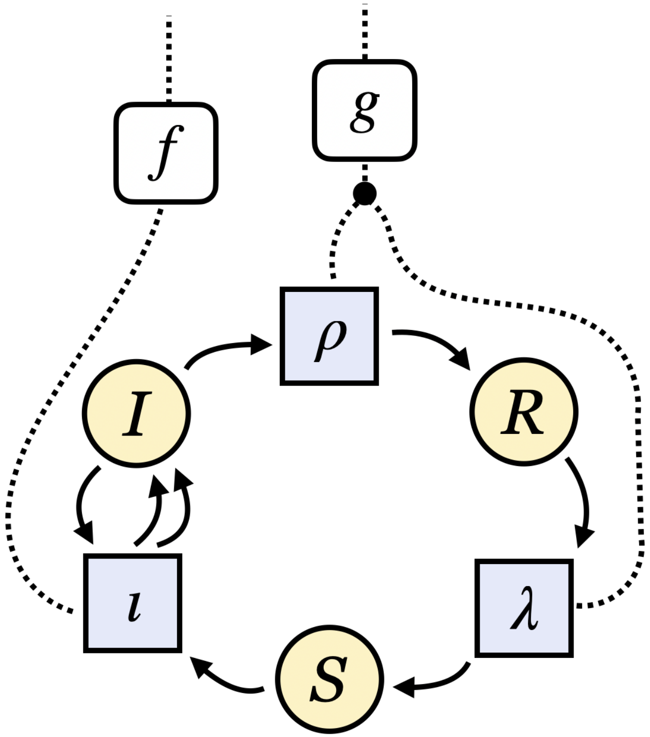
2. 1-cells: $f\colon \mathbb{R}^n_{\geq 0} \to \mathbf{Csp}(\mathsf{Petri}_r)(A, B)$
3. 2-cells: commuting $\varphi\colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}^m_{\geq 0}$

infection $\iota\colon S + I \xrightarrow{r_\iota} 2I$

recovery $\rho\colon I \xrightarrow{r_\rho} R$

loss $\lambda\colon R \xrightarrow{r_\lambda} S$



$$\frac{dS}{dt} = r_\lambda R - r_\iota SI$$

$$\frac{dI}{dt} = r_\iota SI - r_\rho I$$

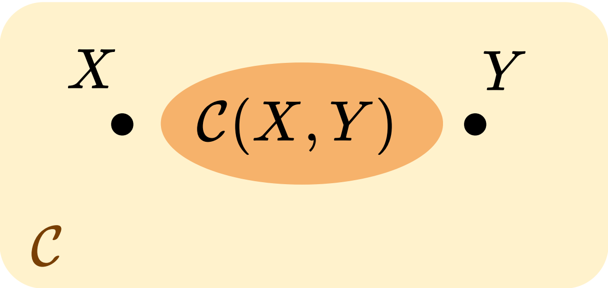
$$\frac{dR}{dt} = r_\rho I - r_\lambda R$$

$$r_\iota = f(x)$$

$$r_\rho = r_\lambda = g(y)$$

We can transfer hypergraph structure via $\iota\colon f \mapsto [* \mapsto f]$.

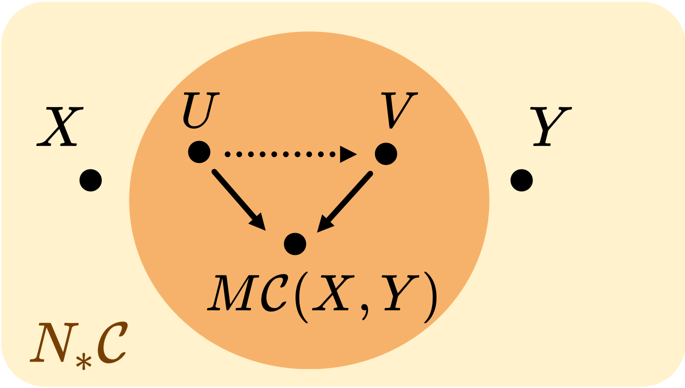
Combining monadic and parametric uncertainty



(symmetric) monoidal
$\mathcal{W}$-category

$$N : \mathcal{V} \xrightarrow{L_M} \mathsf{Kl}_M \xrightarrow{\mathsf{Kl}_M/(-)} \mathsf{cat}$$

change-of-base



(symmetric) monoidal
2-category

Proposition: *(standard)*

The Kleisli category Kl_M of a symmetric monoidal monad (M, η, μ, ∇) is symmetric monoidal.

The identity-on-objects functor $\boldsymbol{L_M : \mathcal{V} \to \mathsf{Kl}_M}$ mapping $f \mapsto f \circ \eta$ ***is strict symmetric monoidal***.

Parametrized distributions of design problems

View DP as Pos-enriched and change-of-base:

$$N\colon \mathsf{Pos} \xrightarrow{\Sigma} \mathsf{Meas} \xrightarrow{L_{\mathsf{D}}} \mathsf{Kl}_{\mathsf{D}} = \mathsf{Stoch} \xrightarrow{\mathsf{Kl}_{\mathsf{D}}/(-)} \mathsf{cat}$$

The symmetric monoidal 2-category $N_*\mathsf{DP}$ has

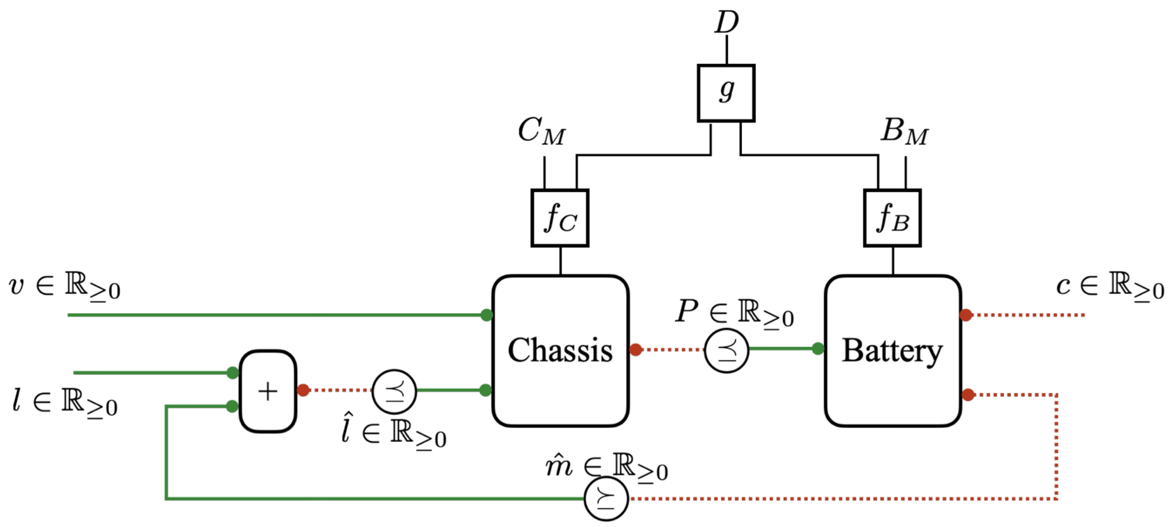
1. objects: Posets,
2. 1-cells: Markov kernels $f\colon U \to \mathsf{DP}(X, Y)$,
3. 2-cells: commuting Markov kernels $\varphi\colon U \to V$.

Transfer compact closed structure via

$$\iota\colon f \mapsto [* \mapsto \delta_f]$$

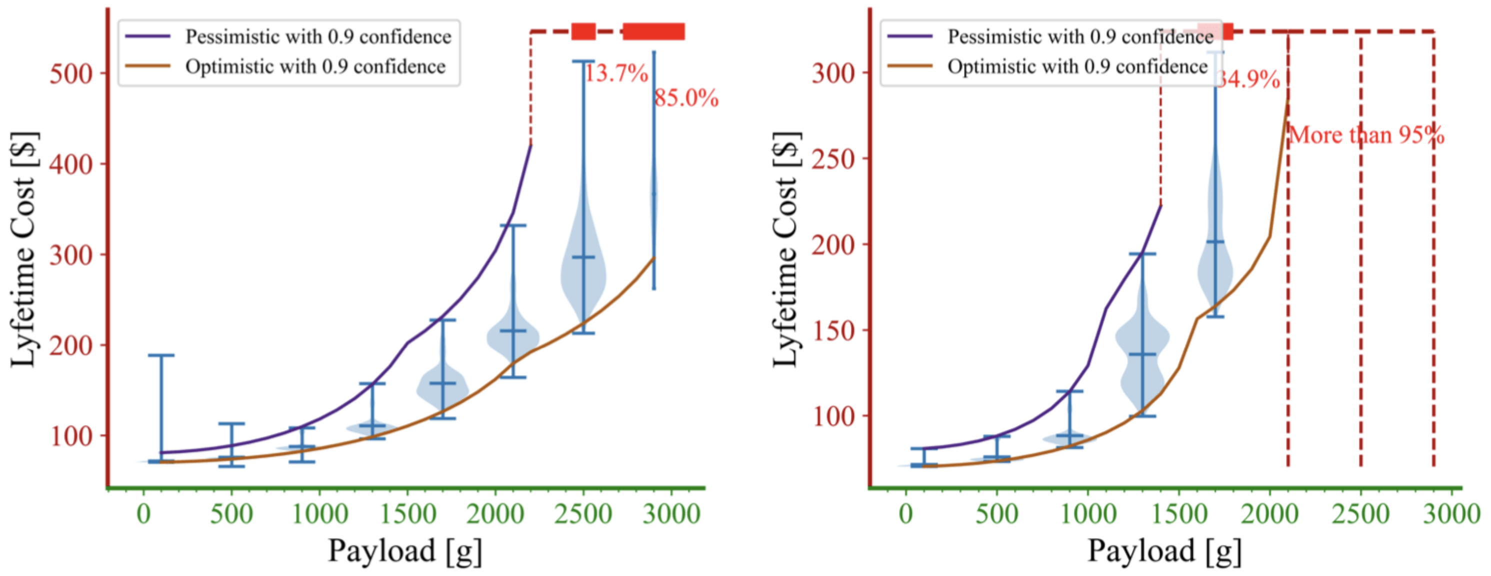
Applications:

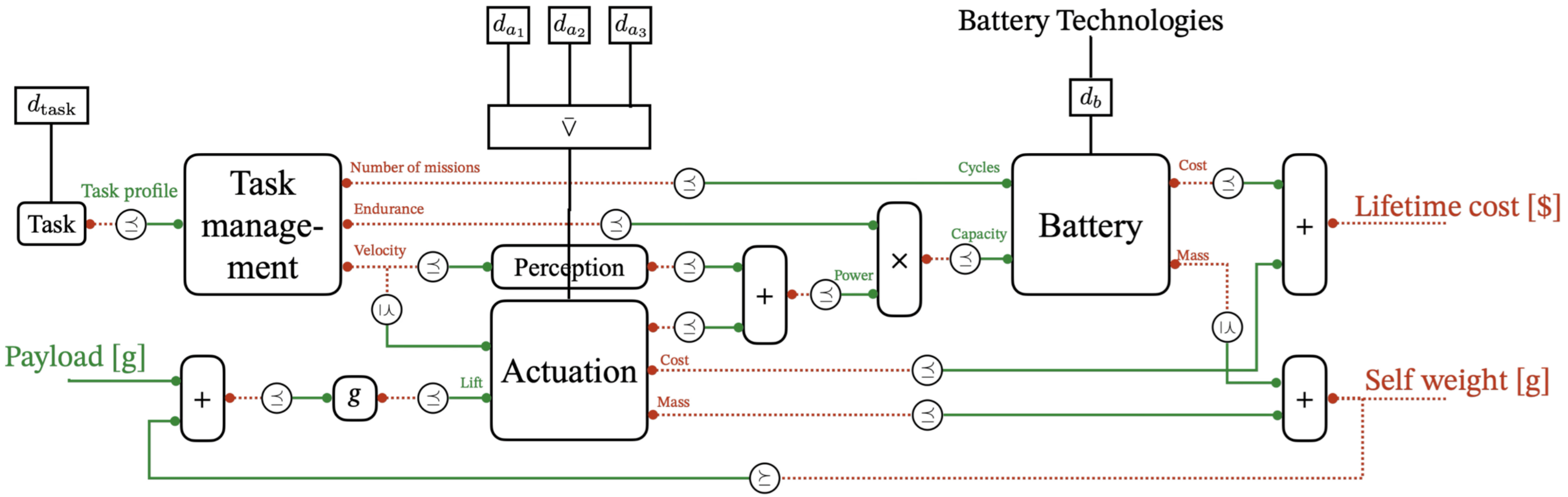
1. Sensitivity / robustness
2. Learn design problems from data
3. Active learning with surrogates



Markov kernel $D \otimes C_M \otimes B_M \to \mathsf{DP}(\mathbb{R}^2_{\geq 0}, \mathbb{R}_{\geq 0})$.

Parametrized distributions of design problems





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